# Coordinated Science Laboratory

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# SYNTHESIS OF THRESHOLD NETWORKS BY ALOGIC FUNCTIONS

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Corrections for Report R-124.

Page 2, Eq. 1: $g_1$ should be $g_i$

Page 11, Example under Eq. 21:

ax should be 2x, $2x\bar{y}\bar{z}$ should be $x\bar{y}\bar{z}$, and $+0\bar{x}\bar{y}\bar{z}$ should be $-\bar{x}\bar{y}\bar{z}$.

Page 12, Example:

$x_1(1-x_2)(1-x_3)$ should be $x_1(1-\bar{x}_2)(1-\bar{x}_3)$

$x_1 - (1-\bar{x}_1)\bar{x}_2 - (1-\bar{x}_1)\bar{x}_3 - \bar{x}_1\bar{x}_2$ should be

$x_1 - (1-\bar{x}_1)\bar{x}_2 - (1-\bar{x}_1)\bar{x}_3 - \bar{x}_1\bar{x}_2 - \bar{x}_1\bar{x}_3$

Page 14, 3rd line from the top: $W(2x_1+x_2+x_2-1)$ should be $W(2x_1+x_2+x_3-1)$.

Page 15, 7th line from the bottom: One of $x_1(1-x_2)(1-x_3)$ should be removed.

Page 20, 15th line from the top: $F_1 = F_2 \cap f_p$ should be $F_1 = F_2 \cup f_p$.

Page 20, 18th line from the top: $\alpha_i$ in the Equation should be $\gamma_i$.

Page 2o, 5th line from the top: $F_1 \cap f_p$ should be $F_1 \cap \bar{f}_p$.

Page 20, 5th line from the bottom: $\bigcup_{j=1}^{r} f_j$ should be $\bigcup_{j=1}^{\gamma} f_j$.

Page 26, 2nd line from the top: g(x) should be g(X).

Page 26, 4th line from the top: $\sum \alpha_i x_i = k$ should be $\sum \alpha_i X_i = k$.

## ABSTRACT

By the use of functions called alogic functions consisting of logic operations as well as algebraic operations, the synthesis of threshold or majority networks becomes simple especially for the realization of a switching function by a single threshold or a single majority element. The first section introduces a particular threshold element called a W-element and shows that W-elements and scalar multipliers are sufficient to represent any threshold or majority networks.

The second section shows the properties of alogic functions and clarifies that the synthesis of threshold or majority networks is equivalent, to obtain a suitable alogic function from a switching function.

The third section gives a rather simple method for the realization of a switching function by a single threshold or a single majority element.

An n-dimensional cube associated with n switching variables and n-1 dimensional hyper-planes has been used by others for the investigation of the properties of threshold or majority elements. In the last section, it is shown that an alogic function of the form $\Sigma \alpha_i x_i - k$ represents an n-1 dimensional hyper-plane.

The extension of applications of alogic functions to synthesis of sequential circuits is promising.

CONTENTS

## INTRODUCTION

As the application of majority elements and threshold elements such as parametrons, tunnel diodes, logic circuits, etc.[1,2,3,4] advances, it is desirable to have a method of synthesis of threshold or majority networks. Investigation of conditions for realizability of a switching function by a single threshold or a single majority element[5-10] is a popular problem in this field at present. However, no one has found a simple useful condition for it. This paper, which is an extension of the introductory paper "Alogic Functions and Their Applications,"[11] gives not only a method for realizing a switching function by a single threshold or a single majority element but also gives a synthesis technique for multi-threshold or majority elements networks by using a new functional representation of such networks.

The output $g_o$ of a threshold or a majority element depends on the sum of input signals which indicates that the functional representation of the output $g_o$ of the element should contain algebraic operations. However, an input function applied to each input terminal of the element is usually represented by a switching function or a logic function which suggests that the functional representation of output $g_o$ should also contain logic operations. Hence, a function representing output $g_o$ of a threshold or a majority element as a function of input variables which are switching variables used in this paper consists not only of algebraic operations but also logic operations. Such a function is called an alogic function. Then, the synthesis of a threshold or a majority metwork is to transform a switching function to a suitable alogic function.

An alogic function of the form $\Sigma \alpha_i x_i - k$ where k and $\alpha_i$ are real numbers for all i represents a network consisting of a single threshold or a single majority element. Hence to synthesize a switching function by a single threshold or a single majority element is to obtain such an alogic function from a given switching function. A simple method of obtaining such an alogic function is shown in Section 3.

## 1. TWO BASIC ELEMENTS

A new element called a W-element is introduced here. This with a scaler multiplier K, whose output signal is equal to the product of an input signal and a real number K, will simplify the synthesis of switching function by threshold elements or majority elements. A W-element and a scaler multiplier are called "two basic elements" and it will be seen that any threshold element and majority element can be considered as the combination of the two basic elements.

A W-element shown in Figure 1 consists of m input and one output terminals. On each input terminal, a signal represented by $g_i$ is applied where $g_i$ takes either a real number $a_i$ or zero. Then the output signal $g_o$ of the element is defined to be:

$$g_o = W\left(\sum_{i=1}^{m} g_1\right) = \begin{cases} 1 \text{ if } g = \sum\limits_{i=1}^{m} g_1 > 0 \\ 0 \text{ otherwise} \end{cases} \qquad (1)$$

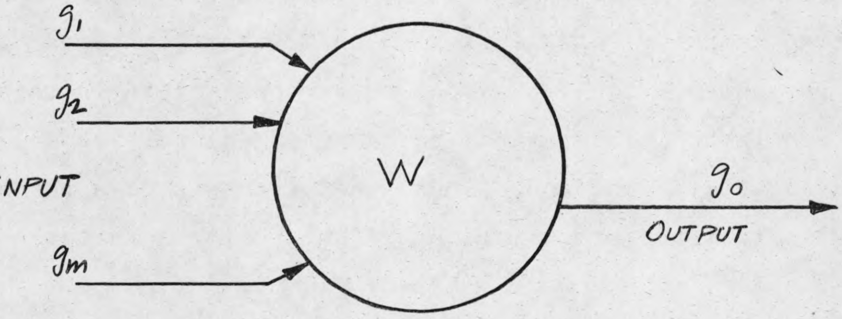


Figure 1.

$W(\sum_{i=1}^{m} g_i)$ is used to indicate the output signal of a W-element whose input signals are $g_1, g_2, \ldots, g_i$. By definition, whenever the value $\sum_{i=1}^{m} g_i$ inside of the parenthesis is larger than zero $W(\sum_{i=1}^{m} g_i)$ is one, otherwise $W(\sum_{i=1}^{m} g_i)$ is defined to be zero. It is clear that a W-element is a particular threshold element whose threshold value is not zero but $\epsilon > 0$ which is smaller than the minimum among the values of $g = \sum_{i=1}^{m} g_i > 0$ with all possible combinations of $g_i = \alpha_i$, 0 (i = 1, 2, ..., m). The symbol $\Sigma g_i$ means the algebraic summation of the values of $g_i$ for i = 1, 2, ... . Since the symbol "+" is used to represent algebraic summation, instead of using Boolean representation, logic representation of switching functions will be employed in this paper, that is, the symbols "$\cup$" union and "$\cap$" intersection will be the operations associated with switching functions. However, the variables in a switching function which are called switching variables are defined to take either 1 or 0, and consequently a switching function takes either 1 or 0.

An input signal $g_i$ is a function of the switching variables but not itself necessarily a switching function. However, since $g_i$ takes either a real number $\alpha_i$ or zero, $g_i$ can be written as $g_i = \alpha_i\, g_i'$ where $g_i'$ takes either 1 or 0. It can be seen that a function, which takes either 1 or 0, whose independent variables are only switching variables, can be expressed by a switching function. Hence, by defining $\alpha_i f_i$ where $f_i$ is a switching function:

$$\alpha_i f_i = \begin{cases} \alpha_i \text{ if } f_i = 1 \\ 0 \text{ otherwise} \end{cases} \quad (2)$$

$g_i$ can be expressed as:

$$g_i = \alpha_i f_i \quad (3)$$

With this definition, a W-element in Figure 1 can be represented as shown in Figure 2 where $f_i$ is a switching function, a real number $\alpha_i$ represents a multiplier for i = 1, 2, ..., m.

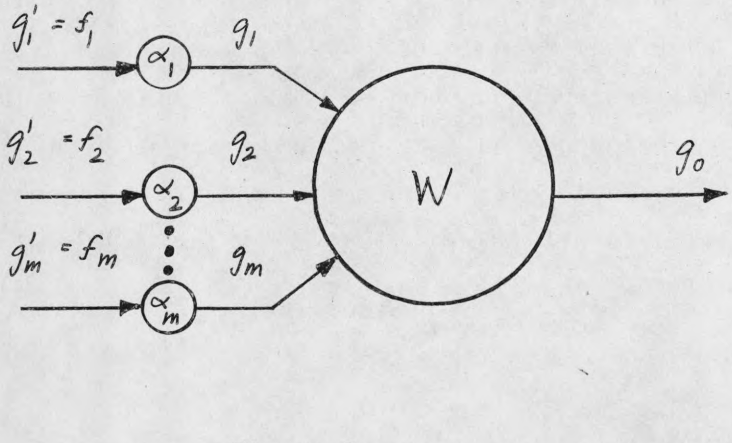


Figure 2.

It must be noticed that $g_i = f_i$ does not mean that $g_i$ is of the form of a switching function, but it does mean that whenever $g_i = 1$, $f_i = 1$ and whenever $g_i = 0$, $f_i = 0$.

The product of $\alpha_i f_i$ defined by Equation (2) can be considered algebraic multiplication of two numbers $\alpha_i$ and $f_i$ if one considers $f_i$ as a variable in a real number field. Because function $g = \Sigma g_i$ contains algebraic summation, g consists of four operations, ($\cup$) union, ($\cap$) intersection, ($\cdot$) algebraic multiplication, and (+) algebraic summation. A function which consists only of switching variables as independent variables, real numbers as constants, and contains some or all of four operations--union, intersection, algebraic summation and algebraic multiplication--is named an "alogic function". Function g discussed previously is an alogic function. A switching function is also an alogic function by this definition.

The symbols x, y, ..., will be used to represent switching variables, f and F indicate switching functions and $\alpha$, $\beta$, ..., will be real numbers. The use of these symbols makes it possible to eliminate the symbols for intersection and algebraic multiplication in an equation. For example: $\alpha\beta xy$ means $\alpha \cdot \beta \cdot (x \cap y)$.

It will be shown that the synthesis of a switching function by either threshold elements or majority elements is equivalent to the transformation of a switching function to an alogic function. Before discussing the properties of alogic functions, it will be important to show that W-elements and scaler multiplier $\alpha$ can be considered as two basic elements in the synthesis of threshold networks and majority networks. In other words, any threshold element and majority element can be constructed by these two basic elements.

An element shown in Figure 3 which is similar to one shown in Figure 1 except that input signal $G_i$ (i = 1, a, ..., m) takes either $\alpha_i$ or $\beta_i$ ($\alpha_i$ and $\beta_i$ are real numbers and $\alpha_i > \beta_i$) and the threshold value h of the element may not be $\epsilon$ as in the case of a W-element.

The output $G_o$ is defined as:

$$G_o = \begin{cases} \alpha_o & \text{if } \sum_{i=1}^{m} G_i \geqq h \\ \beta_o & \text{otherwise} \end{cases} \qquad (4)$$

This is the definition of a threshold element which is generally accepted.

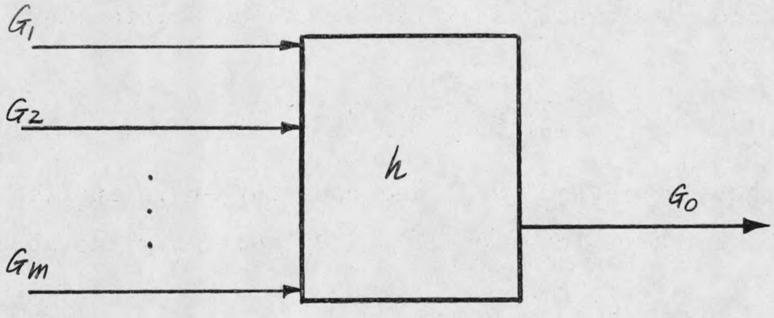


Figure 3.

Since $G_i$ takes either $\alpha_i$ or $\beta_i$, the equation:

$$\sum_{i=1}^{m} G_i \geqq h \tag{5}$$

can be written as:

$$\sum_{i=1}^{m} G'_i + \sum_{i=1}^{m} \beta_i \geqq h \tag{6}$$

or

$$\sum_{i=1}^{m} G'_i + (\sum_{i=1}^{m} \beta_i - h) + \epsilon > 0 \tag{7}$$

where $G'_i$ takes either $\alpha_i - \beta_i$ or zero and $\epsilon > 0$ is a real number which is smaller than or equal to the minimum of $\sum_{i=1}^{m} G_i - h > 0$ with $G_i = \alpha_i, \beta_i$ for $i = 1, 2, \ldots, m$. This means that a threshold element in Figure 3 is equivalent to a W-element with input signals $G'_i$ and one constant input signal whose value is $\sum_{i=1}^{m} \beta_i - h + \epsilon$. Since output $G_o$ will be connected to inputs of another threshold element, $G_o$ is also assumed to take either $\alpha'_o > 0$ or zero which is equivalent to the output $g_o$ of a W-element with a scaler multiplier $\alpha'_o$. Changing a W-element to a general threshold element can also be accomplished in a similar way.

A majority element is another important element for the realization of switching functions. However, a majority element and a W-element are very similar which will be shown next.

An equation which represents a W-element can be written as

$$\sum_{i=1}^{m} g_i - k > 0 \tag{8}$$

where $g_i$ takes either $a_i$ or zero value and constant $-k$ is applied to an input terminal of the element (see Equation (7)). By modifying the above equation, one can obtain

$$\sum_{i=1}^{m} g_i' - 2k + \sum_{i=1}^{m} a_i - \epsilon > 0 \tag{9}$$

where $g_i'$ takes $\pm a_i$ and $\epsilon > 0$ which is smaller than the minimum among all possible values of $\Sigma g_i - k > 0$ with $g_i = a_i, 0$ for all i. Because of $\epsilon$, Equation (9) cannot take zero. However, it is clear that any set of values of g's which satisfies Equation (8) will satisfy Equation (9). Furthermore, any set of values of g's which does not satisfy Equation (8) will not satisfy Equation (9). Hence, output $g_o$ of the majority element shown in Figure 4 represented by Equation (9) and output $g_o$ of the W-element represented by Equation (8) can be expressed by the same switching function. This shows the relationship between W-elements and majority elements. Notice that if every $a_i$ is an integer, one can set $\epsilon = 1$.

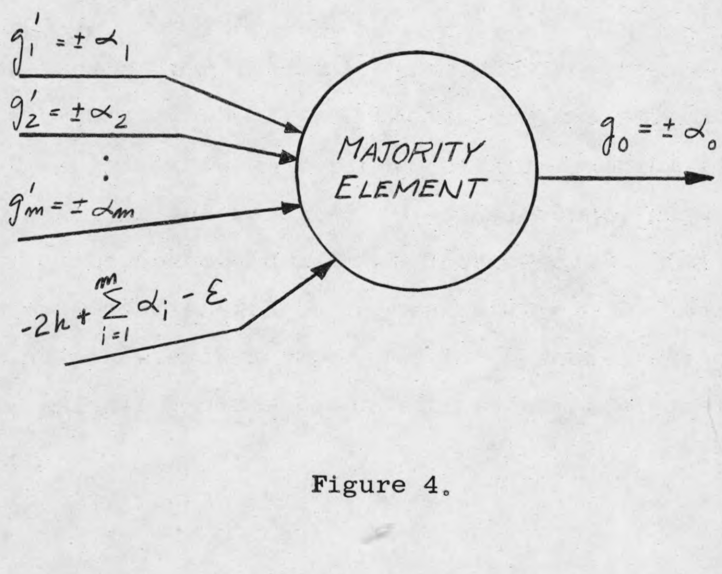


Figure 4.

The above discussion indicates that a W-element and a scaler multiplier are the two basic elements in the synthesis of majority networks and threshold networks.

## 2. ALOGIC FUNCTIONS

Alogic function g which is of the form $\alpha f$ is already defined by Equation 2. In general, an alogic function g is of the form $\sum_{i=1}^{s} \alpha_i f_i$ which is defined as:

$$g = \begin{cases} 0 & \text{if } f_i = 0 \text{ for all } i \\ \sum_{p=1}^{r} \alpha_{i_p} & \text{if } f_{i_1} = f_{i_2} = \dots = f_{i_r} \text{ and all other f's are} \\ & \text{zero for } 0 < r \leqq s \end{cases} \tag{10}$$

where $f_i$ is a switching function*. Fox example, $g = \alpha_1 x_1 + \alpha_2 x_1 x_2 + \alpha_3 x_1 x_3$ takes following values where $f_1 = x_1$, $f_2 = x_1 x_2$, and $f_3 = x_1 x_3$.

$$g = \begin{cases} 0 \text{ if } f_1 = f_2 = f_3 = 0 \\ \alpha_1 \text{ if } f_1 = 1,\ f_2 = f_3 = 0 \\ \alpha_1 + \alpha_2 \text{ if } f_1 = f_2 = 1, \text{ and } f_3 = 0 \\ \alpha_1 + \alpha_3 \text{ if } f_1 = f_3 = 1, \text{ and } f_2 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 \text{ if } f_1 = f_2 = f_3 = 1 \end{cases}$$

Definition: Let $g_1$ and $g_2$ be of the forms $\alpha_1 f_1$ and $\alpha_2 f_2$ respectively. Then the product $g_1 g_2$ is defined to be of the form $g_1 g_2 = \alpha_1 \alpha_2 f_1 f_2$.

For convenience, this operation is called "multiplication". This is algebraic multiplication if one considers switching functions $f_1$ and $f_2$ as variables in real number field. From the above definition, if $g_i$ is of the form $\alpha_i f_i$ for i = 1, 2, and 3, one can obtain the following equalities:

$$g_1(g_2 + g_3) = g_1 g_2 + g_1 g_3 \tag{11}$$

$$(g_1 + g_2) g_3 = g_1 g_3 + g_2 g_3 \tag{12}$$

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*It is already mentioned that switching functions are a particular form of alogic functions. However, "switching functions" are used in this paper to identify these from other alogic functions whenever necessary.

Even if $g_i$ is of the form $\sum_{j=1}^{M_i} \alpha_{ij} f_{ij}$ for i = 1, 2, and 3, Equations 11 and 2 will hold. Furthermore,

$$g_1 + (g_2 + g_3) = (g_1 + g_2) + g_3,\ g_1\ (g_2\ g_3) = (g_1\ g_2)\ g_3 \tag{13}$$

$$g_1 + g_2 = g_2 + g_1,\ g_1\ g_2 = g_2\ g_1 \tag{14}$$

$$g + 0 = g,\ g \cdot 1 = g \tag{15}$$

$$g + (-g) = 0 \tag{16}$$

$$\alpha\ (g_1 + g_2) = \alpha\ g_1 + \alpha\ g_2 \tag{17}$$

and

$$(\alpha + \beta)\ g = \alpha\ g + \beta\ g \tag{18}$$

From the property of switching functions, the following equalities can be obtained where f is a switching function:

$$f^2 = ff = f \tag{19}$$

$$1 + f = f + 1 = 1 \tag{20}$$

It is known that union of switching functions is well defined. However, in general, union of alogic functions which are not switching functions will have no meaning. Hence, it is not necessary to use parenthesis when an alogic function contains both "+" algebraic summation and "$\cup$" union. For example, $g = x_1 \cup x_2 + x_3 \cup x_4 \cup x_5$ means $g = (x_1 \cup x_2) + (x_3 \cup x_4 \cup x_5)$. On the other hand, $\bigcup_i W(g_i)$ is varied for any alogic function $g_i$ because $W(g_i)$, which takes either 1 or 0 by definition, can be considered as a switching variable. This also means that W(g) is an alogic function since a switching function is an alogic function.

Definition: The complement $\bar{g}$ of an alogic function g is equal to $\alpha$ - g if $g = \alpha f$ (where f is a switching function). The complement of g is undefined if $g \neq \alpha f$.

The complement $\bar{f}$ of a switching function f will be 1 - f, if f is considered as an alogic function. The complement $\overline{W(g_i)}$ of $W(g_i)$ is $1 - W(g_i)$ where $g_i$ is any alogic function because $W(g_i) = \mathbf{F}$. On the other hand, if $\bar{g}$ is the complement of g, then $W(\bar{g})$ is the complement of $W(g)$. This means that $1 - W(\bar{g}) = \overline{W(g)}$. In particular, $1 - W(\bar{f}) = \overline{W(f)}$. This does not contradict the definition for the following reasons: Since $\bar{f} = 1 - f$, $W(\bar{f}) = W(1 - f) = W(1) - W(f) = 1 - W(f) = \overline{W(f)}$. Notice that $W(f_1 - f_2) = W(f_1) - W(f_2)$ if $f_1 > f_2$ (which means that whenever $f_2 = 1$, $f_1 = 1$). It can be seen from the definition in the previous section that $W(\alpha f) = W(f)$ if $\alpha > 0$. Hence, $\overline{W(g)} = W(\bar{g})$ is true if there exists the complement of g.

## 3. EQUIVALENT ALOGIC FUNCTIONS

Suppose switching function F is of the form $f_1 \cup f_2$. Then alogic function g which is obtained from F by changing "$\cup$" union to "+" algebraic summation as $f_1 + f_2$ will take 1 or 0 except when both $f_1$ and $f_2$ are 1 in which case g takes the value 2. Hence, $g = F$ or $f_1 + f_2 = f_1 \cup f_2$ if $f_1 f_2 = 0$ (which is equivalent to say that whenever $f_1 = 1$, $f_2 = 0$ and whenever $f_2 = 1$, $f_1 = 0$). Notice that "Whenever $f_1\ (x_1 x_2 \ldots x_n) = 0$, $f_2\ (x_1 x_2 \ldots x_n) = 0$" means that any set of values $e_1, e_2, \ldots,$ and $e_n$ ($e_i = 1, 0$ for $i = 1, 2, \ldots n$) which are given to switching variables $x_1, x_2, \ldots$ and $x_n$ respectively by which $f_1\ (e_1 e_2 \ldots e_n) = 0$ will give $f_2\ (e_1 e_2 \ldots e_n) = 0$. However, "$f\ (x_1 x_2 \ldots x_n) = 0$" means that any set of values $e_1, e_2, \ldots,$ and $e_n$ ($e_i = 1, 0$ for $i = 1, 2, \ldots, n$) which are given to $x_1, x_2, \ldots,$ and $x_n$ respectively will make $f\ (e_1 e_2 \ldots e_n) = 0$.

In general, the alogic function g of the form $\Sigma f_i$ can be obtained from a switching function F such that $g = F$ by changing any "$\cup$" union in the disjunctive canonical form* of F to "+" algebraic summation. It has been shown in Section 1 that the purpose of using alogic functions is to synthesize switching functions by networks consisting of W-elements and scaler multipliers. Hence, to obtain an alogic function g from a given switching function F which has the property that $W(g) = F$ is the main purpose of this paper. Obviously

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*The disjunctive canonical form of $F\ (x_1 x_2 \ldots x_n)$ is of the form $\bigcup_{e=(0 \ldots 0)}^{e=(1\ 1 \ldots 1)} F\ (e_1 e_2 \ldots e_n)\ x_1^{(e_1)}\ x_2^{(e_2)} \ldots x_n^{(1_n)}$ where $e_i$ is either 1 or 0, $x_i^{(1)} = x_i$, $x_i^{(0)} = \bar{x}_i$ and $0 \cdot f = 0$ for any switching function by definition.

above theorem. This is true simply because whenever $F = 1$, g which is obtained by the above theorem takes value larger than zero. Hence, by definition of $W(g)$, $F = W(g)$.

Suppose switching function $f_p$ is of the form that is intersection of switching variables and contains switching variables x and its complement $\bar{x}$. Then $f_p = x\bar{x} f_q = x(1 - x) f_q = 0$ with any set of values $e_1, e_2, \ldots,$ and $e_n$ ($e_i = 1, 0$ for $i = 1, 2, \ldots, n$) which are given to switching variables $x_1, x_2, \ldots,$ and $x_n$ in $f_p$ respectively, where $f_q$ is obtained from $f_p$ by removing x and $\bar{x}$, because $x \cdot x = x$ and $x - x = 0$ by Equation (16).

It is clear that alogic function $W(F)$ is equal to switching function F. Also, if alogic function g is equal to F, then g can be replaced by $W(g)$. In other words, if g takes either 1 or 0, g can be replaced by $W(g)$. For example, suppose $g = x_1 + x_2 x_3$. Then g can be written as $g = x_1 + W(x_2x_3)$. This simple equivalence is very powerful for the synthesis of switching functions. The example at the beginning of this section to obtain the network in Figure 5 uses this relationship.

The theorems and the relationships discussed previously are sufficient to obtain any alogic functions which are equivalent to a given switching function, that is, by these theorems and relationships, and threshold networks (and majority networks) which satisfy a specified switching function can be obtained. However, it may take a long calculation to obtain a simple alogic function. For example, to obtain a simple alogic function $g = x_1 + x_2 + x_3 - 2$ from switching function $F = x_1x_2x_3$, one may use the following procedure:

(1) $g_1 = x_1x_2x_3$ and $g_2 = x_1x_2x_3 - \bar{x}_1\bar{x}_2x_3 - \bar{x}_1x_2\bar{x}_3 - x_1\bar{x}_2\bar{x}_3 - 2\bar{x}_1\bar{x}_2\bar{x}_3$ are equivalent to each other by Theorem 1.

(2) Because $\bar{x} = 1 - x$,

$$\begin{aligned} g_2 &= x_1(1 - x_2)(1 - x_3) - (x_3 + \bar{x}_3)\bar{x}_1\bar{x}_2 - (x_2 + \bar{x}_2)\bar{x}_1\bar{x}_3 - x_1\bar{x}_2\bar{x}_3 \\ &= x_1 - x_1\bar{x}_2 - x_1\bar{x}_3 - \bar{x}_1\bar{x}_2 - \bar{x}_1\bar{x}_3 \\ &= x_1 - (1 - \bar{x}_1)\bar{x}_2 - (1 - \bar{x}_1)\bar{x}_3 - \bar{x}_1\bar{x}_2 = x_1 - \bar{x}_2 - \bar{x}_3 \\ &= x_1 + x_2 + x_3 - 2 \end{aligned}$$

This example indicates the necessity of methods which will simplify the procedure

This indicates that the optimum synthesis is to obtain an alogic function $g_q$ from an alogic function $g_p$ (which may be equal to a given switching function) such that $W(g_q) = W(g_p)$ and the network represented by $g_q$ is better than that represented by $g_p$ in the sense of the optimum whatever one defines. Hence, it is important to study the methods of obtaining such alogic functions.

Definition: Alogic functions $g_p$ and $g_q$ are said to be equivalent to each other or $g_p$ is an equivalent alogic function of $g_q$ if

$$W(g_p) = W(g_q) \tag{21}$$

In the previous example, $g_1$, and $g_2$ are equivalent alogic functions of g.

Definition: The disjunctive canonical form of an alogic function g is of the form $\Sigma \alpha_i f_i$ where $f_i$ is in $\bigcup_{(i)} f_i$ which is the disjunctive canonical form of a switching function $F = W(g)$.

For example, the disjunctive canonical form of $g = ax + y + z - 1$ is $g = 3xyz + 2x\bar{y}\bar{z} + 2xy\bar{z} + 2x\bar{y}z + \bar{x}yz + 0\bar{x}\bar{y}z + 0\bar{x}y\bar{z} + 0\bar{x}\bar{y}\bar{z}$ because the canonical form of $F = W(g) = x \cup yz$ is $F = xyz \cup x\bar{y}\bar{z} \cup xy\bar{z} \cup x\bar{y}z \cup \bar{x}yz \cup \bar{x}\bar{y}z \cup \bar{x}y\bar{z} \cup \bar{x}\bar{y}\bar{z}$

Theorem 1: Let the disjunctive canonical forms of alogic functions $g_p$ and $g_q$ be $\Sigma \alpha_{p_i} f_{p_i}$ and $\Sigma \alpha_{q_i} f_{q_i}$ respectively where $f_{p_i} = f_{q_i}$. Then $g_p$ and $g_q$ are equivalent to each other if and only if for every $\alpha_{p_i}$ and $\alpha_{q_i}$ the following conditions are satisfied:

$$\text{(1)} \quad \text{if } \alpha_{p_i} > 0, \text{ then } \alpha_{q_i} > 0$$

and

$$\text{(2)} \quad \text{if } \alpha_{p_i} \leqq 0, \text{ then } \alpha_{q_i} \leqq 0$$

The proof is obvious by the definition of W(g).

Theorem 2: Alogic function g obtained from switching function F, which is of the form $\bigcup_i f_i$, where $f_i$ $(i = 1, 2, \ldots)$ is of the form that is intersection of switching variables, by changing any union in F to algebraic summation has the property that $W(g) = F$.

Suppose $F = f_1 \cup f_2 \cup f_3 \cup f_4$. Then $g_1 = f_1 + f_2 \cup f_3 \cup f_4$, $g_2 = f_1 \cup f_2 + f_3 \cup f_4$, $g_3 = f_1 + f_2 + f_3 \cup f_4$, etc. are equivalent alogic functions of F by the

For example, suppose $W(x_1) + W(x_2 + x_3 - 1)$ which is equivalent to $F = x_1 \cup x_2x_3$ is given. Then because the above two conditions are satisfied with $K_1 = 1$, $W(x_1) + W(x_2 + x_3 - 1)$ is equal to $W(2x_1 + x_2 + x_2 - 1)$ or $F = W(2x_1 + x_2 + x_3 - 1)$ with $K = 2$. The proof can easily be seen by comparing the right and the left side of Equation (20) under these two conditions.

Theorem 5: Let alogic function g be of the form $g = \sum_{i=1}^{n} a_i x_i - k$ where $k \geqq 0$.

(a) If min $g > 0$ is p, $g_1 = g - s$ is equivalent to g where $0 \leqq s < 0$.

(b) If max $g \leqq 0$ is -p, $g_1 = g + r$ is equivalent to g where $0 \leqq r \leqq p$.

(c) If min $g(x_j = 1) > 0$ is p, $g_1$ which is obtained by changing $a_j$ to $a_j - s$ is an equivalent alogic function of g where $0 \leqq s < p$ and $1 \leqq j \leqq n$.

(d) If max $g(x_j = 1) \leqq 0$ is -p, $g_1$ which is obtained from g by changing $a_j + r$ is an equivalent alogic function of g where $0 \leqq r \leqq p$ and $1 \leqq j \leqq n$.

(e) Alogic functions g and Kg are equivalent to each other for any real number $K > 0$.

For example, if $g = 2x + 2y + 2z - 3$, then max $g \leqq 0$ is -1. Hence, an equivalent alogic function $g_1$ will be $g_1 = 2x + 2y + 2z - 2$. Since min $g_1(x = 1) > 0$ is 2, $g_2$ which is equivalent to $g_1$ can be $g_2 = x + 2y + 2z - 2$. The proof is as follows: Let E be the set of values $e_1, e_2, \ldots$ ($e_i = 1$, 0 for all i) which are given to switching variables $x_1, x_2 \ldots$ in alogic function g.

Since min $g_1 > 0$ is equal to min $g > 0$ minus s where $1 \leqq s < p$ and min $g > 0$ is p, whenever $g > 0$, $g_1 > 0$. Likewise, max $g_1 \leqq 0$ is less than max $g \leqq 0$ which means that whenever $g \leqq 0$, $g_1 \leqq 0$ for $s \neq 0$. Thus g and $g_1$ are equivalent to each other which proves a. The proofs for the others can be accomplished by similar procedure.

Example 1: $F = x_1 \cup x_2x_3 \cup x_2x_4$ can be synthesized by the following procedure to obtain a simple network:

$$\begin{aligned} g &= x_1 + x_2x_3 + x_2x_4 \\ &= x_1 + W(x_2x_3) + W(x_2x_4) = x_1 + W(x_2 + x_3 - 1) + W(x_2 + x_4 - 1) \\ &= x_1 + W[W(x_2 + x_3 - 1) + W(x_2 + x_4 - 1)] \\ &= x_1 + W(2x_2 + x_3 + x_4 - 2) \end{aligned}$$

$$g = W[W(x_1) + W(2x_2 + x_3 + x_4 - 2)]$$

$$= W(3x_1 + 2x_2 + x_3 + x_4 - 2)$$

The network represented by the above alogic functions is shown in Figure 6.

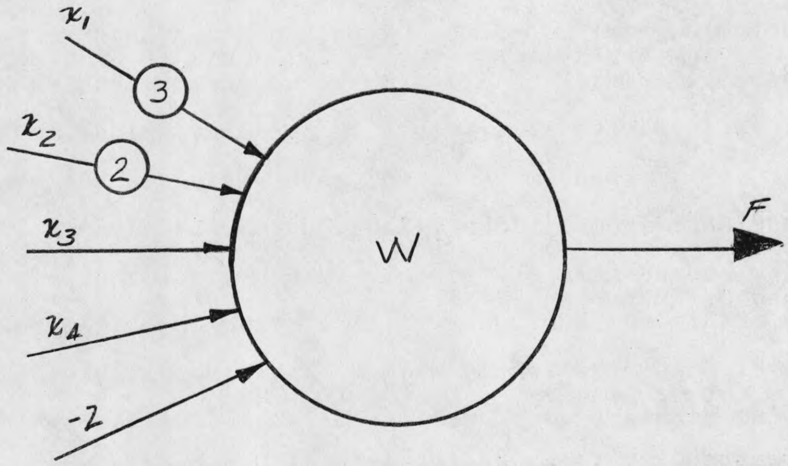


Figure 7.

Example 2: $F = x_1 x_2 x_3 \cup x_1 \bar{x}_2 \bar{x}_3 \cup \bar{x}_1 x_2 \bar{x}_3 \cup \bar{x}_1 \bar{x}_2 x_3$ can be realized by following procedure:

$$g_1 = F = x_1 x_2 x_3 + x_1(1 - x_2)(1 - x_3) + x_1(1 - x_2)(1 - x_3)$$
$$+ (1 - x_1)\, x_2(1 - x_3) + (1 - x_1)(1 - x_2)\, x_3$$

$$= x_1 + x_2 + x_3 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3 - 2x_1 x_2 x_3)$$

$$= x_1 + x_2 + x_3 - 2(x_1 x_2 \bar{x}_3 + x_1 \bar{x}_2 x_3 + x_2 x_3)$$

$$= x_1 + x_2 + x_3 - 2W(g_2)$$

where $g_2 = x_1 x_2 \bar{x}_3 + x_1 \bar{x}_2 x_3 + x_2 x_3$. (This is true because $g_2$ takes either 1 or zero.)

$$W(g_2) = W[W(x_1 + x_2 + \bar{x}_3 - 2) + W(x_1 + \bar{x}_2 + x_3 - 2) + W(x_2 + x_3 - 1)]$$

by Theorem 3.

$$W(g_2) = W[W(x_1 + x_2 + \bar{x}_3 - 2) + W[W(x_1 + \bar{x}_2 + x_3 - 2) + W(x_2 + x_3 - 1)]$$

$$= W[W(x_1 + x_2 + \bar{x}_3 - 2) + W(x_1 + x_2 + 3x_3 - 3)]$$

by Theorem 4. Since

$$W(x_1 + x_2 + 3x_3 - 3) = W[3(x_1 + x_2 + 2x_3 - 2)] = W(x_1 + x_2 + 2x_3 - 2) ,$$

$$W(g_2) = W[W(x_1 + x_2 + \bar{x}_3 - 2) + W(x_1 + x_2 + 2x_3 - 2)]$$

$$= W(3x_1 + 3x_2 + 3x_3 - 5) = W(x_1 + x_2 + x_3 - 1)$$

by Theorems 4 and 5. Hence,

$$g_1 = x_1 + x_2 + x_3 - 2W(x_1 + x_2 + x_3 - 1)$$

The network corresponding to $g_1$ is shown in Figure 8.

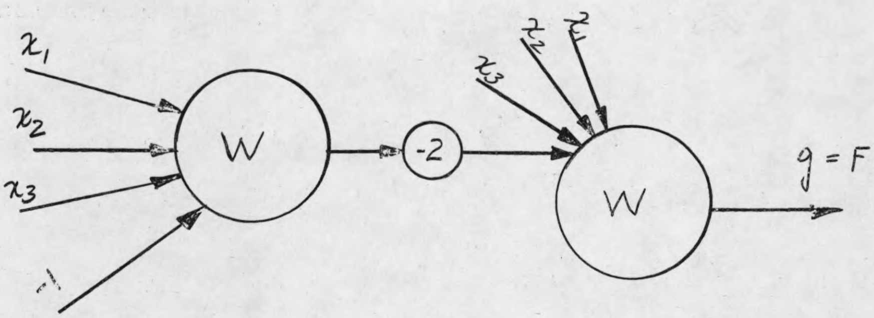


Figure 8.

## 4. SYNTHESIS OF A SINGLE W-ELEMENT CIRCUIT

Linear programming is a suitable mathematics for the synthesis of a single majority element from a switching function. However, the number of simultaneous equations increases exponentially with the increase in the number of switching variables in a switching function. Hence, even with an automatic digital computer, it is not a suitable method. S. Muroga and others[10] show a method of reducing the number of simultaneous equations corresponding to a switching function such that the synthesis of the function by a single majority element by the use of linear programming is practical. O. Stram[7] gives another method of synthesizing a switching function by a single threshold element network which does not have any advantage over the above method.

The synthesis of networks which usually contain more than one W-element by the use of alogic function shown previously can also be useful in the synthesis of a single element network. It is obvious that there exists an alogic function representing a single W-element network which is equivalent to a given switching function if it is realizable by a single majority element (or a single threshold element). Also it is theoretically possible to obtain such an alogic function by the use of Theorems 1 and 2, and simple relationships among equivalent alogic functions discussed previously. The simple process of obtaining an alogic function which represents a single W-element network which will be discussed in this section is accomplished mainly by successive application of Theorem 4 as follows: From a switching function F of the form $\bigcup_{(i)} f_i$, where $f_i$ $(i = 1, 2, \ldots)$ is of the form that is intersection of switching variables, alogic function $g_o$ of the form $\Sigma\, W(f_i)$ can be obtained. Notice that $W(g_o) = F$ or $g_o$ and F are equivalent to each other. Then each $W(f_i)$ is changed to equivalent alogic function of the form $W(\sum_{j=1}^{k+1} x_j - k_i)$ by Theorem 3. Now, the use of Theorem 4 with the help of Corallary 1 and Theorem 5 makes it possible to combine these $W(\sum_{j=1}^{k_i+1} x_j - k_i)$ in $g_1 = \Sigma\, W(\sum_{j=1}^{k_i+1} x_j - k_i)$ one by one until an alogic function of the form $W(\Sigma\, a_i x_i - k)$ is obtained where $k \geqq 0$. For example, if $F = xy \cup xz \cup yz$, $g_o$ will be $g_o = W(xy) + W(xz) + W(yz)$. Then $g_1 = W(x + y - 1) + W(x + z - 1) + W(y + z - 1)$ by Theorem 3. Since $g_2 = W[W(x + y - 1) + W(x + z - 1)] + W(y + z - 1)$ is equivalent to $g_1$ by Corallary 1 and since $W[W(x + y - 1) + W(x + z - 1)] = W(2x + y + z - 2)$ by Theorem 4,

$g_2 = W(2x + y + z - 2) + W(x + z - 1)$. Again, by Theorem 4, $g_3 = W[W(2x + y + z - 2) + W(x + z - 1)] = W(2x + 2y + 2z - 3)$ which represents a single W-element network.

Suppose alogic function $g_1$ is of the form $\sum_{i=1}^{n} \alpha_i x_i - k_1$ and suppose min $g_1 > 0$ is $r > 0$. By multiplying $1/r$ to $g_1$, $g_2 = \sum_{i=1}^{n} \beta_i x_i - k_2$ is obtained where $\beta_i = \alpha_i/r$ and $k_2 = k_1/r$. Notice that $g_1$ and $g_2$ are equivalent to each other. However, min $g_2 > 0$ is 1. Let min $g_2 > 0$ occur with $x_1 = x_2 = \ldots = x_m = 1$ and $x_{m+1} = \ldots = x_n = 0$, that is

$$\sum_{i=1}^{m} \beta_i - k_2 = 1 \tag{21}$$

By multiplying $g_2$ by n, one can obtain another equivalent alogic function $g_3$

$$g_3 = \sum_{i=1}^{n} n\beta_i x_i - n\,k_2 \tag{22}$$

Since min $g_3 > 0$ is n, one can substract $(n - m)$ from $g_3$ by Theorem 5 to obtain an equivalent alogic function $g_4$ as:

$$g_4 = \sum_{i=1}^{n} n\beta_i x_i - n\,k_2 - (n - m) \tag{23}$$

Then min $g_4 > 0$ is m and the max $g_4 \leq 0$ is at most $-(n - m)$.

Consider the alogic function $g'$ obtained from $g_4$ by removing 1 from every coefficient of $x_1$, $x_2$, ..., and $x_m$ and adding 1 to every coefficient of $x_{m+1}$, $x_{m+2}$, ..., and $x_m$ as

$$g' = \sum_{i=1}^{m} (n\beta_i - 1)\, x_i + \sum_{j=m+1}^{n} (n\beta_j + 1)\, x_j - nk_2 - (n - m) \tag{24}$$

which can be written as:

$$g' = \sum_{i=1}^{m} n\beta_i x_i + \sum_{j=m+1}^{n} \beta_j x_j - nk_2 - \sum_{i=1}^{m} x_i + \sum_{j=m+1}^{n} x_j - (n - m)$$

$$g' = g_3 - \sum_{i=1}^{m} x_i - \sum_{j=m+1}^{n} \bar{x}_j = g_3 - (n - \epsilon) - \left[\sum_{i=1}^{m} x_i + \sum_{j=m+1}^{n} \bar{x}_j - (n - \epsilon)\right]$$

$$= g_3 - (n - \epsilon) - g_p \qquad (25)$$

where $0 < \epsilon \leqq 1$. Since min $g_3 > 0$ is n, $g_3 - r$ is equivalent to $g_3$ for $0 \leqq r < n$ by Theorem 5. Also $\sum_{i=1}^{m} x_i + \sum_{j=m+1}^{n} \bar{x}_j$ varies from zero to n - 1 unless $x_1 = x_2 = \ldots = x_m = \bar{x}_{m+1} = \bar{x}_n = 1$ in which case $\sum_{i=1}^{m} x_i + \sum_{j=m+1}^{n} \bar{x}_j$ takes n. Hence, g' is equivalent to $g_3$ unless $x_1 = x_2 = \ldots = x_m = \bar{x}_{m+1} = \ldots = \bar{x}_n = 1$. It is clear that $g_3 - (n - \epsilon)$ is equivalent to $g_3$. Also $W(g_p) = x_1 x_2 \ldots x_m \bar{x}_{m+1} \ldots \bar{x}_n$. Let $F = W(g_4)$ and the disjunctive canonical form of F be $\bigcup_{i=1}^{\nu} f_i$ where every $f_i (1 \leqq i \leqq \nu)$ corresponds to a set of values of x's which gives $g_4 > 0$. Also let $f_p = x_1 x_2 \ldots x_m \bar{x}_{m+1} \ldots x_n$ where $1 \leqq p \leqq \nu$. Then $F' = W(g') = F \cap \bar{f}_p$ which is equal to F except $f_p$ being absent. This gives the following theorem:

Theorem 6: If a switching function F is realizable by a single W-element (or a single threshold or majority element), then there exists a switching function $f_p (1 \leqq p \leqq \nu)$ in the disjunctive canonical form $\bigcup_{i=1}^{\nu} f_i$ of F such that $F' = F \cap \bar{f}_p$ is also realizable by a single W-element.

By Theorem 6, one can say that if a switching function F is realizable by a single W-element network. Then there exists at least r alogic functions $g_1, g_2, \ldots,$ and $g_r$ all of which are of the form $\Sigma a_i x_i - k$ and there exists an order $f_1, f_2, \ldots, f_r$ in the disjunctive canonical form of F such that $W(g_1) = f_1$, $W(g_2) = f_1 \cup f_2, \ldots, W(g_{r-1}) = f_1 \cup f_2 \cup \ldots \cup f_{r-1}$, and $W(g_r) = F$. This theorem also indicates a necessary condition that on switching function is realizable by a single W-element. The next important problem is to find an order of combining these f's in a given switching function F of the form $\bigcup f_i$ by the method described at the beginning of this section.

Suppose $g_1$ and $g_2$ are equivalent to each other. Then the alogic function $g_3 = \xi_1 g_1 + \xi_2 g_2$ is equivalent to $g_1$ and $g_2$ for any real numbers $\xi_1, \xi_2 > 0$. This is clearly true because any set of values of x's which makes $g_1 > 0$ will make $g_2 > 0$ and thus make $\xi_1 g_1 + \xi_2 g_2 > 0$. Furthermore any set of values of x's which makes $g_1 \leqq 0$ will make $g_2 \leqq 0$ which gives $\xi_1 g_1 + \xi_2 g_2 \leqq 0$.

Suppose alogic functions $g_1$ and $g_2$ are of the form

$$\sum_{i=1}^{n} a_i x_i - k_1$$

and

$$\sum_{i=1}^{n} \beta_i \, x_i - k_2$$

respectively where $k_1, k_2 \geqq 0$. Let the disjunctive canonical form of $F_1 = W(g_1)$ be $\bigcup_{i=1}^{\nu} f_i$ where the set of values of x's representing $f_i$ gives $g_1 > 0$ for $i = 1, 2, \ldots, \nu$. Also let $W(g_2) = F_2 = F_1 \bigcap f_p$ ($F_2$ is obtained by removing $f_p$ from $F_1$). Then any set of values of x's which makes both $g_1$ and $g_2$ larger than zero will make $\xi_1 \, g_1 + \xi_2 \, g_2 > 0$ for any real numbers $\xi_1, \xi_2 > 0$. This means that any set of values of x's representing $f_j (1 \leqq j \leqq \nu, \, j \neq p)$ will make $\xi_1 \, g_1 + \xi_2 \, g_2 > 0$. Also any set of values of x's which makes both $g_1$ and $g_2$ equal or less than zero will make $\xi_1 \, g_1 + \xi_2 \, g_2 \leqq 0$ for any real numbers $\xi_1, \xi_2 > 0$. Let $f_p = x_1 \, x_2 \ldots x_m \, x_{m+1} \ldots x_n$. Also let $g_1(x_1 = x_2 = \ldots = x_m = 1, x_{m+1} = \ldots = x_n = 0) = r_1 > 0$ and $g_2(x_1 = x_2 \ldots = x_m = 1, x_{m+1} = \ldots = x_n = 0) = -r_2 < 0$. Then $r_2 \, g_1 + r_1 \, g_2 = 0$ with $x_1 = x_2 = \ldots = x_m = 1$ and $x_{m+1} = \ldots = x_n = 0$. This means that if $F_1$ and $F_2$ are realizable by a single W-element, and if $F_1 = F_2 \bigcap f_p$ where $f_p$ is in the canonical form $f_1 \cup f_2 \cup \ldots \cup f_p \cup \ldots \cup f_r$ of $F_1$, then there exists an alogic function g of the form

$$\sum_{i=1}^{n} \alpha_i \, x_i - k$$

such that $W(g) = F_2$ and $g = 0$ when $f_p = 1$. Hence one can synthesize $F_1$ from g and

$$W\left[\sum_{i=1}^{m} x_i + \sum_{j=m+1}^{n} \bar{x}_j - (n - \epsilon)\right] = f_p$$

by the use of Theorem 4.

Let $g_1 = \sum_{i=1}^{n} \alpha_i x_i - k_1$ and $g_2 = \sum_{i=1}^{n} \beta_i x_i - k_2$ where $k_1, k_2 > 0$. Let the disjunctive canonical form of $F_1 = W(g_1)$ be $\bigcup_{i=1}^{\nu} f_i$ where any set of values of x's corresponding to $f_i (i = 1, 2, \ldots, \nu)$ gives $g_1 > 0$. Also let the disjunctive canonical form of $F_2 = W(g_2)$ be $\bigcup_{j=1}^{\nu} f_j$ where any set of values of x's corresponding to $f_j (j = 1, 2, \ldots, \gamma)$ gives $g_2 > 0$ and $\gamma < \nu$. Notice that $F_2$ is obtained by removing $f_{\gamma+1}, f_{\gamma+2}, \ldots,$ and $f_\nu$ from $F_1$. It is clear that any set of values of x's which makes both $g_1$ and $g_2$ larger than zero will make $\xi_1 \, g_1 + \xi_2 \, g_2 > 0$ for any real numbers $\xi_1, \xi_2 > 0$. Also any set of values of x's which makes both $g_1$ and $g_2$ equal or less than zero will make $\xi_1 g_1 + \xi_2 g_2 \leqq 0$ for $\xi_1, \xi_2 > 0$. Let the values of $g_1$ and $g_2$ with the set of values of x's corresponding to $f_r$ be $V_{1r}$ and $-V_{2r}$ for $r = \gamma + 1, \gamma + 2, \ldots, \nu$ where $V_{1r} > 0$ and $V_{2r} \geqq 0$. Suppose $V_{2\gamma+1}/V_{1\gamma+1}$ is the smallest among all $V_{2r}/V_{1r}$ for $r = \gamma + 1, \gamma + 2, \ldots, \nu$. Then $V_{1\gamma+1}\, g_1 + V_{2\gamma+1}\, g_2$ takes zero with the set of values of x's corresponding to $f_{\gamma+1}$ and takes zero or negative values with sets of values of x's corresponding to $f_{\gamma+2}, f_{\gamma+3}, \ldots$ and $f_\nu$. This indicates the existence of al alogic function $g_3$ of the form $\Sigma\, \theta_i\, x_i - k$ which is equivalent to $g_2$ and has the property that $g_2 = 0$ when $f_{\gamma+1} = 1$. Hence one can combine $W(g_3)$ and $W(g) = f_{\gamma+1}$ by the use of Theorem 4. Also this indicates the following: Suppose a switching function F of the form $\bigvee_{i=1}^{\nu} f_i$ is given. Then one can start from any one of $f_i$ to obtain $g_1$ of the form $\Sigma\, \alpha_i\, x_i - k$ by Theorem 3. Then any of the remaining $f_i = W(g_i)$ obtained by Theorem 3, whichever satisfies the conditions in Theorem 4, can be combined with $g_1$ to form $g_2$ by Theorem 4. (It may be necessary to use Theorem 5 to obtain an equivalent alogic function such that the conditions in Theorem 4 will be fulfilled.) By the previous discussion, it is guaranteed that the above process can be continued until every $f_i$ in F is combined if F is realizable by a single W-element. In other words, suppose $g_q = \sum_{i=1}^{n} \alpha_i\, x_i - k$ is obtained by the above process where $W(g_q) = F_q$ and $F_q \subset F$. Also suppose there exists no switching function $f_j$ in the disjunctive canonical form $\bigcup_{(j)} f_j$ of $F \cap \bar{F}_q$ such that $f_j = W(g_j)$ (by Theorem 3) can be combined with an alogic function of the form $\sum_{i=1}^{n} \beta_i\, x_i - k_2$ which is equivalent to $g_q$. Then by the previous discussion, F is not realizable by a single W-element network.

The method of synthesis which is mentioned above is to combine two alogic functions of the form $g_r = \sum_{i=1}^{n} \alpha_i\, x_i - k_r$ and $g_j = \sum_{i=1}^{n} x_i - (n - \epsilon)$ by the use of Theorem 4 where $0 < \epsilon \leqq 1$. Since $W(g_r)$ usually represents more than one switching function $f_j$ in $\bigvee_{(i)}^{\nu} f_i = F$ where F is a given switchinf function and $W(g_p)$ is equal to $f_p$ $(1 \leqq p \leqq \nu)$, it is impossible to have the situation that $g_p \geqq 0$ whenever $g_r > 0$. In other words, only when $g_r \geqq 0$ whenever $g_p > 0$, one can can combine $W(g_r)$ and $W(g_p)$ together by Theorem 4. Hence if $g_r$ and $g_p$ do not satisfy the conditions in Theorem 4, one must look for an equivalent alogic function $g_r'$ of $g_r$ by Theorem 5 which satisfies the condition that $g_r' \geqq 0$ whenever $g_p > 0$ so that $W(g_r')$ and $W(g_p)$ can be combined. How can one know whether there is an equivalent alogic function of $g_r$ which satisfies the conditions in Theorem 4? Notice that there are infinitely many equivalent alogic functions of $g_r$. For example, $g_r = 3x + 3y + 2z + 2u - 5$ has no equivalent alogic function which is zero when $z = u = 1$ and $x = y = 0$ (that is, $g_p$ is either $z + u - 1$ or $\bar{x} + \bar{y} + z + u - 3$). On the other hand, $3x + y + z - 4$ has an equivalent alogic function which is zero when $y = z = 1$ and $x = 0$. To find a method for testing $g_r$ whether or not there exists an alogic function of $g_r$ satisfying the conditions in Theorem 4 with $g_p$ will be a future problem. However one can always find such an equivalent alogic function $g_r'$ of $g_r$ by the use of Theorem 5 if what is shown by the next theorem exists.

Theorem 6: Suppose $g_p = \sum_{i=1}^{n} \alpha_i x_i - 1$ and $g_q = \sum_{i=1}^{n} \beta_i x_i - 1$ are equivalent to each other when $\alpha_i, \beta_i \geqq 0$ for all i. Then one can obtain $g_q$ from $g_p$ by the use of Theorem 5. For convenience, the proof of the following corollary will be shown before the proof of Theorem 6.

Corollary 2: Suppose $g_\alpha = \sum_{i=1}^{n} \alpha_i x_i - 1$ and $g_\beta = \sum_{i=1}^{n} \beta_i x_i - 1$ are equivalent to each other. Also suppose $\alpha_i \leqq \beta_i$ for all i and $\alpha_i \geqq 0$. Then one can obtain $g_\alpha$ from $g_\beta$ by the use of Theorem 5.

Proof: Without loss of generality, let $\alpha_i < \beta_i$ for $i = 1, 2, \ldots, m$ and $\alpha_i = \beta_i$ for $i = m + 1, m + 2, \ldots n$. Suppose $x_{i_1} = x_{i_2} = \ldots = x_{i_r} = 1$ and all other x's being zero give $\max g_\alpha (x_{i_1} = 1) \leqq 0$ where $1 \leqq i_1 \leqq m$. Then because of $g_\alpha$ and $g_\beta$ are equivalent to each other and $\alpha_{i_1} < \beta_{i_1}$, $g_\beta$ with these values of x must be larger than $\max g_\alpha (x_{i_1} = 1) \leqq 0$. However, $g_\beta$ with these values of x is equal to or less than $\max g_\beta (x_{i_1} = 1) \leqq 0$, that is, $\max g_\alpha (x_{i_1} = 1) \leqq 0$ is not equal to zero. Hence one can increase $\alpha_{i_1}$ by Theorem 5a. The process of increasing $\alpha_i$ which is mentioned above can be continued until every $\alpha_i = \beta_i$. Hence Corollary 2 is true.

With this corollary, Theorem 6 can be proved as follows: Without loss of generality, let $\alpha_i > \beta_i$ for $i = 1, 2, \ldots, r$, $\alpha_i = \beta_i$ for $i = r + 1, r + 2, \ldots, s$ and $\alpha_i < \beta_i$ for $i = s + 1, \ldots, n$. Also let $\min g_p > 0$ be $\epsilon > 0$. Furthermore let $\alpha_i - \beta_i > 0$ be the largest among $\alpha_1 - \beta_1$, $\alpha_2 - \beta_2$, ..., and $\alpha_r - \beta_r$.

With the constant h which satisfies the equation

$$h(\alpha_1 - \beta_1) = \frac{\epsilon}{n + 1}$$

alogic function $g'_p$ of the form

$$g'_p = \sum_{i=1}^{r} [\alpha_i - h(\alpha_i - \beta_i)]x_i + \sum_{j=r+1}^{n} \alpha_j x_j - 1$$

can be obtained where $g'_p$ is equivalent to $g_p$. Notice that $\alpha_i - \beta_i > 0$ for $i = 1, 2, \ldots, r$. Hence every coefficient os $x_i$ of $g'_p$ for $i = 1, 2, \ldots, r$ can be obtained from the corresponding coefficient $\alpha_i$ of $g_p$ by reducing it by $h(\alpha_i - \beta_i)$ which can be done one by one by the use of Theorem 5a.

Consider alogic function $g'_q$

$$g'_q = \frac{1}{1 + \xi} (g_p + \xi g_q)$$

where real number $\xi > 0$. Then it is known that $g'_p$ is equivalent to $g_q$ and every coefficient of $x_i$ in $g'_q$ can be expressed as

$$\frac{\alpha_i + \xi \beta_i}{1 + \xi} = \alpha_i - \frac{\xi}{1 + \xi} (\alpha_i - \beta_i)$$

Hence,

$$\frac{\alpha_i + \xi \beta_i}{1 + \xi} > \alpha_i \text{ if } \alpha_i < \beta_i$$

$$\frac{\alpha_i + \xi \beta_i}{1 + \xi} = \alpha_i \text{ if } \alpha_i = \beta_i$$

and

$$\frac{\alpha_i + \xi \beta_i}{1 + \xi} < \alpha_i \text{ if } \alpha_i > \beta_i$$

By setting $h = \xi/(1 + \xi)$, every coefficient of $g'_p$ is equal to or smaller than that of $g'_q$. Hence by Corollary 2, this process can be continued until $g_q$ is obtained which proves the theorem.

Since alogic functions Kg and q are equivalent to each other for $K > 0$, also since any alogic function of the form $\Sigma \gamma_i x_i - k$ $(k > 0)$ can be changed to

$\Sigma \alpha_i x_i - k'$ where $k' \geqq 0$ and $\alpha_i \geqq 0$ for all i because of the relationship $x = 1 - \bar{x}$, one can state that any alogic function of the form $\Sigma \alpha_i x_i - k$ can be changed to another alogic function of the form $\Sigma \beta_i x_i - k'$ by the use of Theorem 5 if these are equivalent to each other.

Theorem 7: The complement of W(g) where $g = \Sigma \alpha_i x_i - k$ is $W(\epsilon - g)$ where $\epsilon > 0$ is equal or less than min $g > 0$.

Notice that $\epsilon - g$ is not the complement of g unless g takes either $\epsilon$ or zero. However, the theorem is true because whenever $g > 0$ (which is equivalent to saying that whenever $g \geqq 1$), $1 - g \leqq 0$ and whenever $g \leqq 0$, $1 - g > 0$. This theorem indicates the obvious property that if F is realizable by a single W-element, so is $\overline{F}$ and conversely.

Since any alogic function g which is of the form $\sum_{i=1}^{n} \alpha_i x_i - k$ ($\alpha_i$ is a real number for all i), there exists $\epsilon > 0$ which is smaller than min $g > 0$, one can reduce every $\alpha_i$ in g by at least $\epsilon/(n + 1)$ and increase k by at least $\epsilon/(n + 1)$. Also it is true that there always exists a rational number $\beta_i$ and k' where $\alpha$ $\alpha_i \leqq \beta_i \leqq \alpha_i - \epsilon/(n + 1)$ and $k \leqq k' \leqq k + \epsilon/(n + 1)$ which means that there always exists an equivalent alogic function $g' = \sum_{i=1}^{n} \beta_i x_i - k'$ of g where k' and $\beta_i$ are rational numbers for all i. Since $\xi g'$ is equivalent to $g'$ for any positive real number $\xi$, there always exists an equivalent alogic function $g'' = \sum_{i=1}^{n} \theta_i x_i - k''$ of g where $k''$ and $\theta_i$ are integers for all i.

Instead of modifying a final result to obtain an alogic function $g''$ of the form $\sum_{i=1}^{n} \theta_i x_i - k''$ ($k''$ and $\theta_i$ being integers for all i), one can restrict alogic functions, which are used in the synthesis of a switching function by W-elements, to being only those which have the property that every constant appearing in the functions is an integer. To do this, the following modification for Theorems 4, 5, and 7 are necessary in order that k and $\alpha_i$ in an alogic function $g = \sum_{i=1}^{n} \alpha_i x_i - k$ obtained by the use of these theorems will be integers.

(1) In Theorem 4, $K = K' + 1$ is sufficient.

(2) In Theorem 5, s, r, and K must be integers and $\epsilon = 1$.

(3) In Theorem 7, $\epsilon = 1$ should be used.

Finally,

(4) To change integer $\alpha$ in an alogic function $g = \sum_{i=1}^{n} \alpha_i x_i - k$ by Theorem 5 it may be necessary to use (e) before a, b, c, or d in the theorem so that $\alpha$ can be changed to another integer.

Example: From $F = x \cup yz \cup yv$, $g_o$ can be written as $g_o = W(x) + W(yz) + W(yv)$. By Theorem 3, $g_1 = W(x) + W(y + z - 1) + W(y + v - 1)$. By Corollary 1, $g_2 = W[W(x) + W(y + z - 1)] + W(y + v - 1)$ is equivalent to $g_1$. Since x and y + z - 1 will satisfy the conditions in Theorem 4 with K = 1, $g_3$ is equal to $g_3 = W(2x + y + z - 1) + W(y + v - 1)$. Again by Corollary 1, $g_4 = W[W(2x + y + z - 1) + W(y + v - 1)]$ is equivalent to $g_3$. Since $2x + y + z - 1$ and $y + v - 1$ will satisfy the conditions in Theorem 4 with K = 1, $g_5$ which will be the final form is equal to $g_5 = W(4x + 3y + 2z + v - 3)$. By the use of Theorem 5, $g_6$ can be written from $g_5$ as $g_6 = W(4x + 3y + z + v - 3) = F$ because min $g_5$ $(z = 1) > 0$ is 2 and max $g_5$ $(z = 1) \leqq 0$ is 0. The W-element network represented by $g_6$ is shown in Figure 9a and a majority element corresponding to the W-element is shown in Figure 9b.

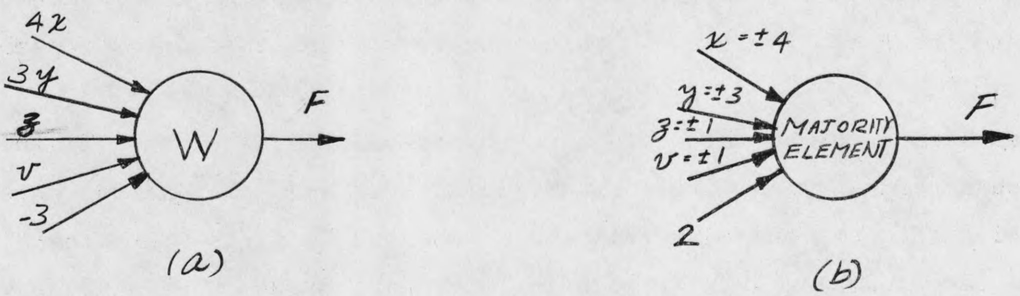


Figure 9.

## 5. FURTHER REMARKS ABOUT ALOGIC FUNCTIONS

Suppose the n-dimensional cube is defined by setting up a coordinate system on the cube with coordinates $(e_1, e_2, \ldots, e_n)$ and by defining $x_j^{e_j} = x_j$ if $e_j = 1$ and $x_j^{e_j} = \bar{x}_j$ if $e_j = 0$ where $e_j = 1, 0$. Then every vertex in the cube represents a combination of n switching variables. Furthermore, let vertex $v_o$ corresponding to $\bar{x}_1 \bar{x}_2 \ldots \bar{x}_n$ be an origin and all edges which are connected to $v_o$ be the axis $X_1, X_2, \ldots$ and $X_n$ of n-dimensional Euclidean space such that $X_i = 1$ corresponds to $e_i = 1$ for all i. Hence the point $X_1 = 1$, $X_2 = X_3 = \ldots = X_n = 0$ is the vertex of the cube representing $x_1 \bar{x}_2 \bar{x}_3 \ldots \bar{x}_n$.

The point $X_1 = X_2 = \ldots = X_k = 1$, $X_{k+1} = X_{k+2} = \ldots = X_n = 0$ is the vertex of the cube representing $x_1 x_2 \ldots x_k \bar{x}_{k+1} \ldots \bar{x}_n$. For convenience, $g(x)$ is defined to be the function obtained from alogic function g by changing every switching variable $x_i$ in g to $X_i$. Then $g(X)$ of alogic function $g = \Sigma a_i x_i - k$ is an $n - 1$ dimensional hyper-plane which cuts the cube into two parts such that every vertex in one of these two parts represents $f_j$ in the disjunctive canonical form $\bigcup_{(i)} f_i$ at $F = W(g)$ and every vertex in the other part including every vertex which is on the plane represents $f_q$ in the disjunctive canonical form $\bigcup_{(q)} f_q$ of $\bar{F}$. The one of these two parts which contains the vertices representing $f_j$ in $\bigcup_{(i)} f_i = F$ is named the F-part of the cube and the other is named the $\bar{F}$-part of the cube. Then $g'(X)$ corresponding to an equivalent alogic function $g'$ of g is also an $n - 1$ dimensional hyper-plane which cuts the cube into the F-part and the $\bar{F}$-parts. To obtain an equivalent alogic function $g'_r$ of $g_r$, which has the property that whenever $g_p > 0$, $g'_r \geqq 0$ where $W(g_p) = f_p$, $f_p$ represents a vertex in the cube is to obtain g hyper-plane $g'_r(X)$ which cuts the cube into the F and $\bar{F}$-parts and which passes through the vertex representing $f_p$.

Theorem 4 is a method to obtain a new hyper-plane $g''(X)$ from $g(X)$ where $g(X)$ passes through vertex v and cuts the cube into the F and the $\bar{F}$-parts where $F = W(g)$ and $g''(X)$ cuts the cube into the $F''$ and the $\bar{F}''$ parts where the $F''$ part of the cube contains all vertices in the F part of the cube and the vertex v. The $\bar{F}''$ part of the cube contains all the vertices in the $\bar{F}$ part of the cube except vertex v.

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